

STABILITY OF HELICOIDS IN HYPERBOLIC THREE-DIMENSIONAL SPACE

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ABSTRACT. For a family of minimal helicoids $\{\mathcal{H}_a\}_{a \geq 0}$ in the hyperbolic 3-space \mathbb{H}^3 (see §1.2 or §2 for detail definitions), there exists a constant $a_c \approx 2.17966$ such that the following statements are true:

- \mathcal{H}_a is a globally stable minimal surface if $0 \leq a \leq a_c$, and
- \mathcal{H}_a is an unstable minimal surface with index one if $a > a_c$.

1. INTRODUCTION

Let $(\overline{M}^3, \overline{g})$ be a 3-dimensional Riemannian manifold (compact or complete), and let Σ be a surface (compact or complete) immersed in \overline{M}^3 . We choose a local orthonormal frame field $\{e_1, e_2, e_3\}$ for \overline{M}^3 such that, restricted to Σ , the vectors $\{e_1, e_2\}$ are tangent to Σ and the vector e_3 is perpendicular to Σ . Let $A = (h_{ij})_{2 \times 2}$ denote the second fundamental form of Σ , whose entries h_{ij} are represented by

$$h_{ij} = \langle \overline{\nabla}_{e_i} e_3, e_j \rangle, \quad i, j = 1, 2,$$

where $\overline{\nabla}$ is the covariant derivative in \overline{M}^3 , and $\langle \cdot, \cdot \rangle$ is the metric of \overline{M}^3 . The immersed surface Σ is called a *minimal surface* if its *mean curvature* $H = h_{11} + h_{22}$ is identically equal to zero.

Let $\overline{\text{Ric}}(e_3)$ denote the Ricci curvature of \overline{M}^3 in the direction e_3 , and $|A|^2 = \sum_{i,j=1}^2 h_{ij}^2$. The *Jacobi operator* on Σ is defined by

$$\mathcal{L} = \Delta_\Sigma + (|A|^2 + \overline{\text{Ric}}(e_3)), \quad (1.1)$$

where Δ_Σ is the Laplacian for the induced metric on Σ .

1.1. Stability of minimal surfaces. Suppose that Σ is a complete minimal surface immersed in a complete Riemannian manifold \overline{M}^3 . For any compact connected subdomain Ω of Σ , its *first eigenvalue* is defined by

$$\lambda_1(\Omega) = \inf \left\{ - \int_\Omega u \mathcal{L} u \, d \text{vol} \mid u \in C_0^\infty(\Omega) \text{ and } \int_\Omega u^2 \, d \text{vol} = 1 \right\}, \quad (1.2)$$

where $d \text{vol}$ denotes the area element on the surface Σ . We say that Ω is *stable* if $\lambda_1(\Omega) > 0$, *unstable* if $\lambda_1(\Omega) < 0$ and *maximally weakly stable* if

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$\lambda_1(\Omega) = 0$. For a complete minimal surface Σ without boundary, it is *globally stable* or *stable* if any compact subdomain of Σ is stable.

Remark 1. It's easy to verify that $\Omega \subset \Sigma$ is stable if and only if

$$\int_{\Omega} |\nabla_{\Sigma} u|^2 d \text{vol} > \int_{\Omega} (|A|^2 + \overline{\text{Ric}}(e_3)) u^2 d \text{vol} \quad (1.3)$$

for all $u \in C_0^{\infty}(\Omega)$, where ∇_{Σ} is the covariant derivative of Σ .

The following result is well known (for example see [10, Lemma 6.2.5]).

Lemma 1.1. *Let Ω_1 and Ω_2 be connected subdomains of Σ with $\Omega_1 \subset \Omega_2$, then*

$$\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) .$$

If $\Omega_2 \setminus \overline{\Omega_1} \neq \emptyset$, then

$$\lambda_1(\Omega_1) > \lambda_1(\Omega_2) .$$

Remark 2. If $\Omega \subset \Sigma$ is maximally weakly stable, then for any compact connected subdomains $\Omega_1, \Omega_2 \subset \Sigma$ satisfying $\Omega_1 \subsetneq \Omega \subsetneq \Omega_2$, we have that Ω_1 is stable whereas Ω_2 is unstable.

Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ be an exhaustion of Σ , then the first eigenvalue of Σ is defined by

$$\lambda_1(\Sigma) = \lim_{n \rightarrow \infty} \lambda_1(\Omega_n) . \quad (1.4)$$

This definition is independent of the choice of the exhaustion. We say that Σ is *stable* if $\lambda_1(\Sigma) > 0$ and *unstable* if $\lambda_1(\Sigma) < 0$.

Let $\Sigma \subset \overline{M}^3$ be a complete minimal surface, and let Ω be any subdomain of Σ . Recall that a *Jacobi field* on Ω is a C^{∞} function ϕ such that $\mathcal{L}\phi = 0$ on Ω . The following theorem was proved by Fischer-Colbrie and Schoen in [6, Theorem 1] (see also [4, Proposition 1.39]).

Theorem 1.2 (Fischer-Colbrie and Schoen). *If Σ is a complete minimal surface in a 3-dimensional Riemannian manifold \overline{M}^3 , then Σ is stable if and only if there exists a positive function $\phi : \Sigma \rightarrow \mathbb{R}$ such that $\mathcal{L}\phi = 0$.*

The *Morse index* or *index* of compact connected subdomain Ω of Σ is the number of negative eigenvalues of the Jacobi operator \mathcal{L} (counting with multiplicity) acting on the space of smooth sections of the normal bundle that vanishes on $\partial\Omega$. The *Morse index* of Σ is the supremum of the Morse indices of compact subdomains of Σ .

1.2. Helicoids in \mathbf{H}^3 and the main theorem. We consider the Lorentzian 4-space \mathbb{L}^4 , i.e. a vector space \mathbb{R}^4 with the Lorentzian inner product

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 \quad (1.5)$$

where $x, y \in \mathbb{R}^4$. The hyperbolic space \mathbf{H}^3 can be considered as the unit sphere of \mathbb{L}^4 :

$$\mathbf{H}^3 = \{x \in \mathbb{L}^4 \mid \langle x, x \rangle = -1, x_1 \geq 1\}. \quad (1.6)$$

The helicoid \mathcal{H}_a is the surface parametrized by the (u, v) -plane in the following way:

$$\mathcal{H}_a = \left\{ x \in \mathbf{H}^3 \mid \begin{array}{ll} x_1 = \cosh u \cosh v, & x_2 = \cosh u \sinh v \\ x_3 = \sinh u \cos(av), & x_4 = \sinh u \sin(av) \end{array} \right\} \quad (1.7)$$

For any constant $a \geq 0$, the helicoid $\mathcal{H}_a \subset \mathbf{H}^3$ is an embedded minimal surface (see [7]). In the hyperboloid model \mathbf{H}^3 , the axis of the helicoid \mathcal{H}_a is given by

$$(\cosh v, \sinh v, 0, 0), \quad -\infty < v < \infty,$$

which is the intersection of the x_1x_2 -plane and the three dimensional hyperboloid \mathbf{H}^3 .

In this paper, we will prove the following theorem.

Theorem 5.2. *For a family of minimal helicoids $\{\mathcal{H}_a\}_{a \geq 0}$ in the hyperbolic 3-space \mathbf{H}^3 given by (1.7), there exist a constant $a_c \approx 2.17966$ such that the following statements are true:*

- (1) \mathcal{H}_a is a globally stable minimal surface if $0 \leq a \leq a_c$, and
- (2) \mathcal{H}_a is an unstable minimal surface with index one if $a > a_c$.

Remark 3. In [7, Theorem 2], Mori proved that \mathcal{H}_a is globally stable if $a \leq 3\sqrt{2}/4 \approx 1.06$ (see also [8, Theorem 5.1]), and that \mathcal{H}_a is unstable if $a \geq \sqrt{105\pi}/8 \approx 2.27$.

Notation. We use the following notations for the 3-dimensional hyperbolic space in this paper: (1) \mathbf{H}^3 denotes the hyperboloid model; (2) \mathbb{B}^3 denotes the Poincaré ball model; and (3) \mathbb{H}^3 denotes the upper half space model. Usually we use \mathbb{H}^3 to denote the hyperbolic 3-space for the most time.

2. HELICOIDS IN HYPERBOLIC SPACE

In order to visualize the helicoids in the hyperbolic 3-space, we will study the parametric equations of helicoids in the Poincaré ball model and upper half space model of the hyperbolic 3-space.

Consider the Poincaré ball model \mathbb{B}^3 , i.e., the unit sphere

$$\mathbb{B}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\},$$

equipped with the hyperbolic metric

$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - r^2)^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The helicoid \mathcal{H}_a in the ball model \mathbb{B}^3 is given by

$$\mathcal{H}_a = \left\{ (x, y, z) \in \mathbb{B}^3 \left| \begin{array}{l} x = \frac{\sinh u \cos(av)}{1 + \cosh u \cosh v} , \\ y = \frac{\sinh u \sin(av)}{1 + \cosh u \cosh v} , \\ z = \frac{\cosh u \sinh v}{1 + \cosh u \cosh v} \end{array} \right. \right\} , \quad (2.1)$$

where $-\infty < u, v < \infty$.

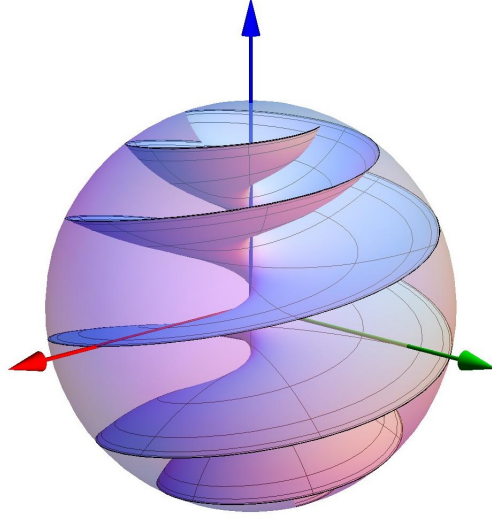


FIGURE 1. The helicoid \mathcal{H}_a with $a = 5$ in the Poincaré ball model of hyperbolic space. The curves perpendicular to the spirals are geodesics in \mathbb{B}^3 .

Consider the upper half space model of hyperbolic 3-space, i.e., a three dimensional space

$$\mathbb{H}^3 = \{z + tj \mid z \in \mathbb{C} \text{ and } t > 0\} ,$$

which is equipped with the (hyperbolic) metric

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2} ,$$

where $z = x + iy$ for $x, y \in \mathbb{R}$. In the upper half space model, the helicoid \mathcal{H}_a is given by (see Figure 2)

$$\mathcal{H}_a = \{(z, t) \in \mathbb{H}^3 \mid z = e^{v+ia v} \tanh u \text{ and } t = e^v \operatorname{sech} u\} , \quad (2.2)$$

and the axis of \mathcal{H}_a is the t -axis, where $-\infty < u, v < \infty$.

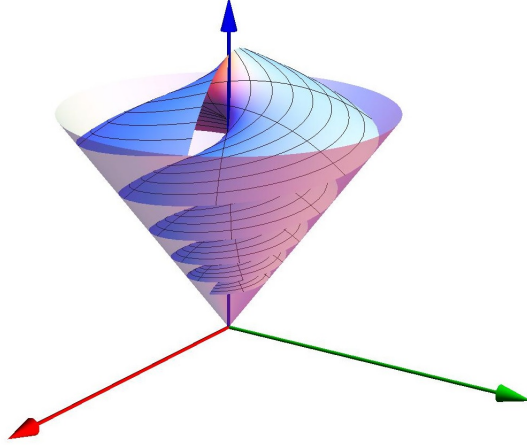


FIGURE 2. The helicoid \mathcal{H}_a with $a = 10$ in the upper half space model of hyperbolic space, where the cone whose vertex is the origin is the log 2-neighborhood of the t -axis. The curves perpendicular to the spirals are geodesics in \mathbb{H}^3 .

Remark 4. To derive the formulas in (2.1) and (2.2), we apply the isometries from the hyperboloid model to the Poincaré ball model and the upper half space model (see [1, §A.1]).

3. CATENOIDS IN HYPERBOLIC 3-SPACE

In this section we define the catenoids (rotation surfaces with zero mean curvature) in hyperbolic 3-space \mathbb{H}^3 (see [5, 9] for detail), since we need these to prove Theorem 5.2.

3.1. Catenoids in the hyperboloid model \mathbf{H}^3 . We follow [5] to describe three types of catenoids in \mathbf{H}^3 . Recall that \mathbb{L}^4 is the Lorentzian 4-space whose inner product is given by (1.5). Its isometry group is $\mathrm{SO}^+(1, 3)$. For any subspace P of \mathbb{L}^4 , let $\mathrm{O}(P)$ be the subgroup of $\mathrm{SO}^+(1, 3)$ which leaves P pointwise fixed.

Definition 3.1. Let $\{e_1, \dots, e_4\}$ be an orthonormal basis of \mathbb{L}^4 (it may not be the standard orthonormal basis). Suppose that $P^2 = \mathrm{span}\{e_3, e_4\}$, $P^3 = \mathrm{span}\{e_1, e_3, e_4\}$ and $P^3 \cap \mathbf{H}^3 \neq \emptyset$. Let C be a regular curve in $P^3 \cap \mathbf{H}^3 = \mathbf{H}^2$ that does not meet P^2 . The orbit of C under the action of $\mathrm{O}(P^2)$ is called a *rotation surface generated by C around P^2* .

If a rotation surface in Definition 3.1 has mean curvature zero, then it's called a *catenoid* in \mathbf{H}^3 . There are three types of catenoids in \mathbf{H}^3 : spherical catenoids, hyperbolic catenoids, and parabolic catenoids.

3.1.1. *Spherical catenoids.* The spherical catenoid is obtained as follows. Let $\{e_1, \dots, e_4\}$ be an orthonormal basis of \mathbb{L}^4 such that $\langle e_4, e_4 \rangle = -1$. Suppose that P^2 , P^3 and C are the same as those defined in Definition 3.1. For any point $x \in \mathbb{L}^4$, write $x = \sum x_k e_k$. If the curve C is parametrized by

$$x_1(s) = \sqrt{\tilde{a} \cosh(2s) - 1/2}, \quad \tilde{a} > 1/2 \quad (3.1)$$

and

$$x_3(s) = \sqrt{x_1^2(s) + 1} \sinh(\phi(s)), \quad x_4(s) = \sqrt{x_1^2(s) + 1} \cosh(\phi(s)), \quad (3.2)$$

where

$$\phi(s) = \int_0^s \frac{\sqrt{\tilde{a}^2 - 1/4}}{(\tilde{a} \cosh(2\sigma) + 1/2)\sqrt{\tilde{a} \cosh(2\sigma) - 1/2}} d\sigma, \quad (3.3)$$

then the rotation surface, denoted by $\mathcal{M}_1^2(\tilde{a})$, is a complete minimal surface in \mathbf{H}^3 , which is called a *spherical catenoid*.

3.1.2. *Hyperbolic catenoids.* The hyperbolic catenoid is obtained as follows. Let $\{e_1, \dots, e_4\}$ be an orthonormal basis of \mathbb{L}^4 such that $\langle e_1, e_1 \rangle = -1$. Suppose that P^2 , P^3 and C are the same as those defined in Definition 3.1. For any point $x \in \mathbb{L}^4$, write $x = \sum x_k e_k$. If the curve C is parametrized by

$$x_1(s) = \sqrt{\tilde{a} \cosh(2s) + 1/2}, \quad \tilde{a} > 1/2 \quad (3.4)$$

and

$$x_3(s) = \sqrt{x_1^2(s) - 1} \sin(\phi(s)), \quad x_4(s) = \sqrt{x_1^2(s) - 1} \cos(\phi(s)), \quad (3.5)$$

where

$$\phi(s) = \int_0^s \frac{\sqrt{\tilde{a}^2 - 1/4}}{(\tilde{a} \cosh(2\sigma) - 1/2)\sqrt{\tilde{a} \cosh(2\sigma) + 1/2}} d\sigma, \quad (3.6)$$

then the rotation surface, denoted by $\mathcal{M}_{-1}^2(\tilde{a})$, is a complete minimal surface in \mathbf{H}^3 , which is called a *hyperbolic catenoid*.

3.1.3. *Parabolic catenoids.* The parabolic catenoid is obtained as follows. Let $\{e_1, \dots, e_4\}$ be a pseudo-orthonormal basis of \mathbb{L}^4 such that $\langle e_1, e_1 \rangle = \langle e_3, e_3 \rangle = 0$, $\langle e_1, e_3 \rangle = -1$ and $\langle e_j, e_k \rangle = \delta_{jk}$ for $j = 2, 4$ and $k = 1, 2, 3, 4$ (see [5, P. 689]). Suppose that P^2 , P^3 and C are the same as those defined in Definition 3.1. For any point $x \in \mathbb{L}^4$, write $x = \sum x_k e_k$. If the curve C is parametrized by

$$x_1(s) = \sqrt{\cosh(2s)}, \quad (3.7)$$

and

$$x_4(s) = x_1(s) \int_0^s \frac{d\sigma}{\sqrt{\cosh^3(2\sigma)}}, \quad x_3(s) = \frac{1 + x_4^2(s)}{x_1(s)}, \quad (3.8)$$

then the rotation surface, denoted by \mathcal{M}_0^2 , is a complete minimal surface in \mathbf{H}^3 , which is called a *parabolic catenoid*. Up to isometries, the parabolic catenoid \mathcal{M}_0^2 is unique (see [5, Theorem (3.14)]).

3.2. Catenoids in the Poincaré ball model \mathbb{B}^3 . We follow [9] to describe the spherical catenoids in \mathbb{B}^3 . Suppose that G is a subgroup of $\text{Isom}^+(\mathbb{B}^3)$ that leaves a geodesic $\gamma \subset \mathbb{B}^3$ pointwise fixed. We call G the *spherical group* of \mathbb{B}^3 and γ the *rotation axis* of G . A surface in \mathbb{B}^3 invariant under G is called a *spherical surface* or a *surface of revolution*.

Now suppose that G is the spherical group of \mathbb{B}^3 along the geodesic

$$\gamma_0 = \{(u, 0, 0) \in \mathbb{B}^3 \mid -1 < u < 1\}, \quad (3.9)$$

then $\mathbb{B}^3/G \cong \mathbb{B}_+^2$, where $\mathbb{B}_+^2 = \{(u, v) \in \mathbb{B}^2 \mid v \geq 0\}$.

For any point $p = (u, v) \in \mathbb{B}_+^2$, there is a unique geodesic segment γ' passing through p that is perpendicular to γ_0 at q . Let $x = d(O, q)$ and $y = d(p, q) = d(p, \gamma_0)$ (see Figure 3), where $d(\cdot, \cdot)$ denotes the hyperbolic distance. It's well known that \mathbb{B}_+^2 can be equipped with the *metric of warped product* in terms of the parameters x and y as follows:

$$ds^2 = \cosh^2 y \cdot dx^2 + dy^2, \quad (3.10)$$

where dx represents the hyperbolic metric on the geodesic γ_0 in (3.9).

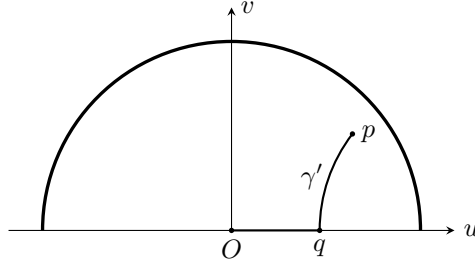


FIGURE 3. For a point p in \mathbb{B}_+^2 with the warped product metric, its coordinates (x, y) are defined by $x = d(O, q)$ and $y = d(p, q)$.

Suppose that \mathcal{C} is a surface of revolution in \mathbb{B}^3 with respect to the geodesic γ_0 , then the curve $\sigma = \mathcal{C} \cap \mathbb{B}_+^2$ is called the *generating curve* of \mathcal{C} . If the curve $\sigma_{\bar{a}} \subset \mathbb{B}_+^2$ is given by the parametric equations:

$$x(t) = \pm \int_{\bar{a}}^t \frac{\sinh(2\bar{a})}{\cosh \tau} \frac{d\tau}{\sqrt{\sinh^2(2\tau) - \sinh^2(2\bar{a})}},$$

and $y(t) = t$, where $\bar{a} > 0$ is a constant and $t \geq \bar{a}$, then the spherical surface generated by $\sigma_{\bar{a}}$, denoted by $\mathcal{C}_{\bar{a}}$, is a complete minimal surface in \mathbb{B}^3 , which is called a *spherical catenoid*.

The following result will be used for proving Theorem 5.2.

Lemma 3.2 (Bérard and Sa Earp). *The spherical catenoid $\mathcal{M}_1^2(\tilde{a})$ defined in §3.1.1 is isometric to the spherical catenoid $\mathcal{C}_{\bar{a}}$, where*

$$2\tilde{a} = \cosh(2\bar{a}). \quad (3.11)$$

Proof. The spherical catenoid $\mathcal{C}_{\bar{a}}$ is obtained by rotating the generating curve $\sigma_{\bar{a}}$ along the axis γ_0 . The distance between $\sigma_{\bar{a}}$ and γ_0 is \bar{a} .

The spherical catenoid $\mathcal{M}_1^2(\tilde{a})$ defined in §3.1.1 can be obtained by rotating the generating curve C given by (3.1) and (3.2) along the geodesic $P^2 \cap \mathbf{H}^3$. The distance between C and $P^2 \cap \mathbf{H}^3$ is $\sinh^{-1}(\sqrt{\tilde{a} - 1/2})$.

Hence the spherical catenoid $\mathcal{M}_1^2(\tilde{a})$ is isometric to the spherical catenoid $\mathcal{C}_{\bar{a}}$ if and only if $\bar{a} = \sinh^{-1}(\sqrt{\tilde{a} - 1/2})$, which implies (3.11). \square

Remark 5. The equation (3.11) can be found in [2, p. 34].

4. CONJUGATE MINIMAL SURFACE

Let $\overline{M}^3(c)$ be the 3-dimensional space form whose sectional curvature is a constant c .

Definition 4.1 ([5, pp. 699-700]). Let $f : \Sigma \rightarrow \overline{M}^3(c)$ be a minimal surface in isothermal parametrs (σ, t) . Denote by

$$I = E(d\sigma^2 + dt^2) \quad \text{and} \quad II = \beta_{11}d\sigma^2 + 2\beta_{12}d\sigma dt + \beta_{22}dt^2$$

the first and second fundamental forms of f , respectively.

Set $\psi = \beta_{11} - i\beta_{12}$ and define a family of quadratic form depending on a parameter θ , $0 \leq \theta \leq 2\pi$, by

$$\beta_{11}(\theta) = \operatorname{Re}(e^{i\theta}\psi), \quad \beta_{22}(\theta) = -\operatorname{Re}(e^{i\theta}\psi), \quad \beta_{12}(\theta) = \operatorname{Im}(e^{i\theta}\psi). \quad (4.1)$$

Then the following forms

$$I_{\theta} = I \quad \text{and} \quad II_{\theta} = \beta_{11}(\theta)d\sigma^2 + 2\beta_{12}(\theta)d\sigma dt + \beta_{22}(\theta)dt^2$$

give rise to an isometry family $f_{\theta} : \widetilde{\Sigma} \rightarrow \overline{M}^3(c)$ of minimal immersions, where $\widetilde{\Sigma}$ is the universal covering of Σ . The immersion $f_{\pi/2}$ is called the *conjugate immersion* to $f_0 = f$.

The following result is obvious, but it's crucial to prove Theorem 5.2.

Lemma 4.2. *Let $f : \Sigma \rightarrow \overline{M}^3(c)$ be an immersed minimal surface, and let $f_{\pi/2} : \widetilde{\Sigma} \rightarrow \overline{M}^3(c)$ be its conjugate minimal surface, where $\widetilde{\Sigma}$ is the universal covering of Σ . Then the minimal immersion f is globally stable if and only if its conjugate immersion $f_{\pi/2}$ is globally stable.*

Proof. Let \tilde{f} be the universal lifting of f , then $\tilde{f} : \widetilde{\Sigma} \rightarrow \overline{M}^3(c)$ is a minimal immersion. It's well known that the global stability of f implies the global stability of \tilde{f} . Actually the minimal surfaces Σ and $\widetilde{\Sigma}$ share the same Jacobi operator defined by (1.1). If Σ is globally stable, there exists a positive Jacobi field on Σ according to Theorem 1.2, which implies that $\widetilde{\Sigma}$ is also globally stable, since the corresponding positive Jacobi field on $\widetilde{\Sigma}$ is given by composing.

Next we claim that \tilde{f} and $f_{\pi/2}$ share the same Jacobi operator. In fact, since the Laplacian depends only on the first fundamental form, \tilde{f} and $f_{\pi/2}$ have the same Laplacian. Furthermore according to the definition of the conjugate minimal immersion, \tilde{f} and $f_{\pi/2}$ have the same square norm of the second fundamental form, i.e. $|A|^2 = (\beta_{11}^2 + 2\beta_{12}^2 + \beta_{22}^2)/E^2$, where we used the notations in Definition 4.1.

By (1.1) and Theorem 1.2, the proof of the lemma is complete. \square

5. STABILITY OF HELICOIDS

For hyperbolic and parabolic catenoids in \mathbf{H}^3 defined in §3.1.2 and §3.1.3, do Carmo and Dajczer proved that they are globally stable (see [5, Theorem (5.5)]). Furthermore, Candel proved that the hyperbolic and parabolic catenoids are least area minimal surfaces (see [3, p. 3574]).

The following result can be found in [5, Theorem (3.31)].

Theorem 5.1 (do Carmo-Dajczer). *Let $f : \mathcal{M}^2 \rightarrow \mathbf{H}^3$ be a minimal catenoid defined in §3.1. Its conjugate minimal surface is the geodesically-ruled minimal surface \mathcal{H}_a given by (1.7) where*

$$\begin{cases} a = \sqrt{(\tilde{a} + 1/2)/(\tilde{a} - 1/2)} , & \text{if } \mathcal{M}^2 = \mathcal{M}_1^2(\tilde{a}) \text{ is spherical ,} \\ a = \sqrt{(\tilde{a} - 1/2)/(\tilde{a} + 1/2)} , & \text{if } \mathcal{M}^2 = \mathcal{M}_{-1}^2(\tilde{a}) \text{ is hyperbolic ,} \\ a = 1 , & \text{if } \mathcal{M}^2 = \mathcal{M}_0^2 \text{ is parabolic .} \end{cases} \quad (5.1)$$

Now we are able to prove the main theorem.

Theorem 5.2. *For a family of minimal helicoids $\{\mathcal{H}_a\}_{a \geq 0}$ in the hyperbolic 3-space \mathbf{H}^3 given by (1.7), there exist a constant $a_c \approx 2.17966$ such that the following statements are true:*

- (1) \mathcal{H}_a is a globally stable minimal surface if $0 \leq a \leq a_c$, and
- (2) \mathcal{H}_a is an unstable minimal surface with index one if $a > a_c$.

Proof of Theorem 5.2. When $a = 0$, \mathcal{H}_a is a hyperbolic plane, so it is globally stable.

According to Theorem 5.1, when $0 < a < 1$, \mathcal{H}_a is conjugate to the hyperbolic catenoid $\mathcal{M}_{-1}^2(\tilde{a})$, where $a = \sqrt{(\tilde{a} - 1/2)/(\tilde{a} + 1/2)}$ by (5.1), and when $a = 1$, \mathcal{H}_a is conjugate to the parabolic catenoid \mathcal{M}_0^2 in \mathbf{H}^3 . Therefore when $0 < a \leq 1$, the helicoid \mathcal{H}_a is globally stable by [5, Theorem (5.5)].

When $a > 1$, \mathcal{H}_a is conjugate to the spherical catenoid $\mathcal{M}_1^2(\tilde{a})$ in \mathbf{H}^3 by Theorem 5.1, which is isometric to the spherical catenoid $\mathcal{C}_{\tilde{a}}$ in \mathbb{B}^3 , where $2\tilde{a} = \cosh(2\bar{a})$ by (3.11) and $a = \sqrt{(\tilde{a} + 1/2)/(\tilde{a} - 1/2)}$ by (5.1). Therefore \mathcal{H}_a is conjugate to the spherical catenoid $\mathcal{C}_{\tilde{a}}$ in \mathbb{B}^3 , where

$$a = \coth(\bar{a}) .$$

By [9, Theorem 3.9], $\mathcal{C}_{\tilde{a}}$ is globally stable if $\bar{a} \geq \bar{a}_c \approx 0.49577$, therefore \mathcal{H}_a is globally stable when $1 < a \leq a_c = \coth(\bar{a}_c) \approx 2.17966$.

On the other hand, if $0 < \bar{a} < \bar{a}_c$, then the spherical catenoid $\mathcal{C}_{\bar{a}}$ is unstable with index 1 (see [9, Theorem 1.2]), therefore \mathcal{H}_a is unstable with index 1 when $a > a_c$. \square

REFERENCES

1. Riccardo Benedetti and Carlo Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin, 1992.
2. Pierre Bérard and Ricardo Sa Earp, *Lindelöf's theorem for catenoids revisited*, ArXiv e-prints (2009).
3. Alberto Candel, *Eigenvalue estimates for minimal surfaces in hyperbolic space*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3567–3575 (electronic).
4. Tobias H. Colding and William P. Minicozzi, II, *A course in minimal surfaces*, Graduate Studies in Mathematics, vol. 121, American Mathematical Society, Providence, RI, 2011.
5. Manfredo do Carmo and Marcos Dajczer, *Rotation hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc. **277** (1983), no. 2, 685–709.
6. Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211.
7. Hiroshi Mori, *On surfaces of right helicoid type in H^3* , Bol. Soc. Brasil. Mat. **13** (1982), no. 2, 57–62.
8. Keomkyo Seo, *Stable minimal hypersurfaces in the hyperbolic space*, J. Korean Math. Soc. **48** (2011), no. 2, 253–266.
9. Biao Wang, *Least area Spherical Catenoids in Hyperbolic Three-Dimensional Space*, ArXiv e-prints (2012).
10. Yuanlong Xin, *Minimal submanifolds and related topics*, Nankai Tracts in Mathematics, vol. 8, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

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